

Syntax versus Semantics

Reinhard Kahle¹ and Wilfried Keller²

- 1 CMA and DM, FCT, Universidade Nova de Lisboa
P-2829-526 Caparica, Portugal
kahle@mat.uc.pt
- 2 Universität des Saarlandes
66123 Saarbrücken, Germany
wilfried.keller@uni-saarland.de

Abstract

We report on the idea to use colours to distinguish syntax and semantics as an educational tool in logic classes. This distinction gives also reason to reflect on some philosophical issues concerning semantics.

1998 ACM Subject Classification F.4.1, F.4.m

Keywords and phrases syntax, semantics, colour, philosophy, teaching

1 Introduction

When the first author attended as a first year student a logic course at a philosophy department, he was wondering why the professor was repeating, after introducing the relation \models , “the same” again but using the symbol ‘ \vdash ’. It was, most likely, the negligence of the student, and not the presentation of the professor, which caused this fundamental misunderstanding. In retrospect, there is one question which might make one wonder: how was it possible to miss one of the most fundamental distinctions in modern logic? On a more mature stage, as third year student, the problem repeated itself on a different level: speaking about Gödel’s first incompleteness theorem with a teaching assistant, he refused to continue the discussion when the question was raised what its meaning is from the perspective of “proper Mathematics”. Again, the distinction of syntax and semantics wasn’t clearly seen and the discussion stalled when the participants did not realize that reference of a term like “proper Mathematics” need to be better specified while entering into a discussion of Gödel’s theorem.

Based on the experiences above, when starting to teach logic courses by himself, the first author took it as a particular challenge to present the problems concerning the distinction of syntax and semantics in a persuasive way. In this paper, we present the chosen solution: the use of colours to distinguish the syntactical and semantical role of logical text. We also report on some of the educational insights one might win with this approach together with certain philosophical questions which surface (again) in this context.

We assume that the reader is familiar with first-order logic and has, at least, an idea of Gödel’s incompleteness theorems. Without formal introduction, we use the standard notations of first-order logic and Peano Arithmetic, using typically Greek letters from the end of the alphabet to denote formulae.

Acknowledgements. The first author was partially supported by the Fundação para a Ciência e a Tecnologia (Portuguese Foundation for Science and Technology) through the projects *The Notion of Mathematical Proof* (PTDC/MHC-FIL/5363/2012) and UID/MAT/00297/2013 (Centro de Matemática e Aplicações). We thank the anonymous referees for helpful comments.



2 First-order logic

Let us go *in media res* and recall the definition of the satisfiability relation \models for first-order logic as given by Barwise in [3, p. 21].

► **Definition 1.** Let \mathfrak{M} be an L-structure. We define a relation

$$\mathfrak{M} \models \varphi[s],$$

(read: the assignment s satisfies the formula φ in \mathfrak{M}) for all assignments s and all formulae φ as follows.

- (i) $\mathfrak{M} \models (t_1 = t_2)[s]$ iff $t_1^{\mathfrak{M}}(s) = t_2^{\mathfrak{M}}(s)$,
- (ii) $\mathfrak{M} \models R(t_1, \dots, t_n)[s]$, iff $(t_1^{\mathfrak{M}}(s), \dots, t_n^{\mathfrak{M}}(s)) \in R^{\mathfrak{M}}$,
- (iii) $\mathfrak{M} \models (\neg\varphi)[s]$ iff not $\mathfrak{M} \models \varphi[s]$,
- (iv) $\mathfrak{M} \models (\varphi \wedge \psi)[s]$ iff $\mathfrak{M} \models \varphi[s]$ and $\mathfrak{M} \models \psi[s]$,
- (v) $\mathfrak{M} \models (\varphi \vee \psi)[s]$ iff $\mathfrak{M} \models \varphi[s]$ or $\mathfrak{M} \models \psi[s]$,
- (vi) $\mathfrak{M} \models (\varphi \rightarrow \psi)[s]$ iff either not $\mathfrak{M} \models \varphi[s]$ or else $\mathfrak{M} \models \psi[s]$,
- (vii) $\mathfrak{M} \models (\exists v.\varphi)[s]$ iff there is an $a \in M$ such that $\mathfrak{M} \models \varphi[s(\frac{a}{v})]$,
- (viii) $\mathfrak{M} \models (\forall v.\varphi)[s]$ iff for all $a \in M$, $\mathfrak{M} \models \varphi[s(\frac{a}{v})]$.

Barwise then states: “There is nothing surprising here. It is just making sure that each of our symbols means what we want it to mean.”

This is certainly correct, but it might be worth to reflect shortly what this definition actually does.¹ Looking, for instance, to the clauses for conjunction and disjunction, it simply “lifts” the symbols ‘ \wedge ’ and ‘ \vee ’ to the *meta-level* using the natural language expressions “and” and “or”. Thus, who understands “and” and “or” is supposed to understand how to obtain the truth value of a formula using ‘ \wedge ’ or ‘ \vee ’. The *meta-level* comes into play, because the natural language terms are outside of the \models relation, while the logical symbols are inside. Nothing surprising here, as said. We will see later in § 6 that the situation for the negation is actually more subtle than it looks (and this is a first educational pitfall!). But the issue we like to start with is unambiguously expressed by Barwise (in continuation of the citation above):

There is one possibly confusing point, in (i), caused by our using $=$ for both the real equality (on the right-hand side) and the symbol for equality (on the left). Many authors abhor this confusion of use and mention and use something like \equiv or \approx for the symbol.

It is this confusion², which will repeat itself when one comes to the language of arithmetic, we like to address. As Barwise remarks, many authors try to resolve the problem by the use of different symbols for syntactical and semantical equality. This would be a way out, but it comes at the price that the formal language appears artificial, in particular, if this

¹ Here a disclaimer is in order. In the following we don’t intend to criticise Barwise’s presentation; first, the citation is taken from a handbook article and not from a textbook (and we may add: from an excellent article in an excellent handbook); secondly, mathematically there is no issue here, the definition is surely the correct one; we are just looking for aspects which are of interest in an educational perspective.

² And it might be doubted, whether the difficulty at hand is best couched in terms of use and mention. It seems to us rather clear, however, that it is educationally not best dealt with by introducing subtleties of quotation theory.

is extended to arithmetical terms. Let us mention the clumsy notation \mathbf{k}_n for the numeral representing the number n in the classical textbook of Shoenfield [15]. With respect to equality, Sernadas and Sernadas [14], for instance, use \cong as special sign on the syntactical side. Cori and Lascar [5] use, quite generally, a bar over the syntactical symbol to refer to its *semantic* interpretation. However, in general (and with good reasons), the syntactic symbols are chosen to match exactly with their intended meaning; the resulting “overloading” is just a problem when the difference of syntax and semantics is the subject under discussion.

Our solution is to use the same symbols, but different colours to distinguish the use of symbols on the syntactical and semantical side. So, let us choose red for syntax and blue for semantics.³ After giving a definition of first-order language with all syntactic expressions in red, and the definition of a structure with blue for the elements of the structure, the definition above reads as follows:

► **Definition 2.** Let \mathfrak{M} be an L-structure. We define a relation

$$\mathfrak{M} \models \varphi[s],$$

(read: the assignment s satisfies the formula φ in \mathfrak{M}) for all assignments s and all formulae φ as follows.⁴

- (i) $\mathfrak{M} \models (t_1 = t_2)[s]$ iff $t_1^{\mathfrak{M}}(s) = t_2^{\mathfrak{M}}(s)$,
- (ii) $\mathfrak{M} \models R(t_1, \dots, t_n)[s]$, iff $(t_1^{\mathfrak{M}}(s), \dots, t_n^{\mathfrak{M}}(s)) \in R^{\mathfrak{M}}$,
- (iii) $\mathfrak{M} \models (\neg\varphi)[s]$ iff not $\mathfrak{M} \models \varphi[s]$,
- (iv) $\mathfrak{M} \models (\varphi \wedge \psi)[s]$ iff $\mathfrak{M} \models \varphi[s]$ and $\mathfrak{M} \models \psi[s]$,
- (v) $\mathfrak{M} \models (\varphi \vee \psi)[s]$ iff $\mathfrak{M} \models \varphi[s]$ or $\mathfrak{M} \models \psi[s]$,
- (vi) $\mathfrak{M} \models (\varphi \rightarrow \psi)[s]$ iff either not $\mathfrak{M} \models \varphi[s]$ or else $\mathfrak{M} \models \psi[s]$,
- (vii) $\mathfrak{M} \models (\exists v.\varphi)[s]$ iff there is an $a \in M$ such that $\mathfrak{M} \models \varphi[s(\frac{a}{v})]$,
- (viii) $\mathfrak{M} \models (\forall v.\varphi)[s]$ iff for all $a \in M$, $\mathfrak{M} \models \varphi[s(\frac{a}{v})]$.

Now, the difference of $=$ and $=$ is obvious; the “possible confusion” will even catch the student’s eye, and this is of particular educational value. In addition, the quantifier clauses allow to distinguish better between the syntactical object variable v and the semantical element a —even if a is here, of course, a meta-variable for “real elements” of M .

Introducing now, as counterpart to the semantic consequence relation, a derivation relation \vdash which exclusively, deals with red objects, syntactic and semantic reasoning is already distinguished by colours. It is our teaching experience, that these colours are of great help for the students, providing some kind of orientation in rather technical proofs like these for the compactness and the completeness theorem. As prime examples for the use of colours we may also mention the Skolem paradox, which, like the completeness theorem, can be regarded as a corollary of the compactness theorem.

And so the question of soundness and completeness is the next one to be addressed. Even in a compact course on Gödel’s incompleteness theorems these results of Gödel’s dissertation

³ The particular choice of the colours is arbitrary; it is, however, known that one should avoid yellow and green for beamer presentations.

⁴ For sure, as one referee pointed put, colouring these expressions does not remove entirely the possibility of misunderstanding: The functional expression $t_1^{\mathfrak{M}}$, for instance, takes the syntactic argument t_1 and evaluates it relative to the semantic model \mathfrak{M} —so the value of the whole expression of course denotes a blue item. The interpretation function itself does not appear in this notation, and therefore, luckily, we do not have to decide on its colour. However, with interested students we had some interesting discussions about its colour.

should be discussed.⁵ Today, they are usually not proven along the lines of Gödel’s original work, but rather with the well-known technique introduced by Henkin. This technique is standard for many completeness proofs for quite a couple of logics and, therefore, constitute an integral tool in the toolbox of the mathematical logician: it proceeds by constructing a blue world (validating the negation of a formula, that cannot be proven in the syntactic calculus) out of the red material itself. (The situation is somewhat dual to the more demanding case of (set-theoretic) forcing, where one, so to speak, bestows properties of a language on certain blue sets, thereby emulating the red world with originally blue objects.)

3 Arithmetic

The use of the colours exhibits its real potential only, when we study Arithmetic and want to motivate, for instance, Gödel’s incompleteness theorems for Peano Arithmetic. In its original form, Gödel’s proof actually entirely refrains from the semantic realm and in some sense requires only “red” reasoning. It, indeed, is a theorem about the red world, but one that operates on a meta-level. It speaks about the axiomatic system, but it does not reason about its meaning, quite like Hilbert’s attempted consistency proofs do. And likewise it uses only finitistically acceptable means. In [6, fn. 1, p. 337] the editors explicitly attribute Gödel’s avoiding of the concept of truth in his 1931 paper to the “respect for the ‘prejudice’ of the Hilbert school.” Further reflections can be found in [7].

But, of course, the first incompleteness theorem can—and probably should—be motivated by the question whether Peano Arithmetic is complete with respect to the *standard model*. Peano Arithmetic may be defined over the set $\{0, S, +, \cdot\}$ of non-logical symbols. The standard model, as semantic object, can be given in blue as follows: $\mathfrak{N} = \langle \mathbb{N}, 0, S, +, \cdot \rangle$. It goes without saying that the red symbols should be interpreted by their respective blue counterparts. Gödel’s first incompleteness theorem tells us, that the red world, as long as a consistent recursive axiomatic system is chosen, will miss some formula *and* its negation⁶, it cannot decide all given formulae by proof. However, the blue world so to speak decides formulae: Either a formula or its negation is satisfied (in a model by a valuation); this is a direct consequence of clause (iii) in Def. 1 (see also §6 below). There is no gap in the blue world—but there will always be one in the (sensibly chosen, i.e., recursive and consistent) red one. This then not only entails that no red world (as above) fixes the blue world of Arithmetic as its only model—this could already be concluded from Löwenheim–Skolem—but that not all red statements that are true about the blue world can be proven in the red realm.

So, *how* are the blue objects in $\mathfrak{N} = \langle \mathbb{N}, 0, S, +, \cdot \rangle$ actually given? \mathbb{N} as the set of natural numbers $\{0, 1, 2, \dots\}$ can be taken for granted, and with it, of course, also the object 0. But what *is* +? It should be, of course, the set of all triples (a, b, c) such that $a + b = c$. But this doesn’t help us, as we just moved the question to another level, writing + in black (this plus sign can, of course, not be blue if we don’t want to run into an immediate circle). To avoid

⁵ In introductory logic courses in philosophy seminars these results will usually not be proven, but, of course, deserve at least to be mentioned and discussed.

⁶ Of course on pain of inconsistency it should *not* prove any formula together with its negation.

it, one could get back to use of dots and say that $+$ is the set:

$$\begin{aligned} &\{(0, 0, 0), (0, 1, 1), (0, 2, 2), (0, 3, 3), \dots \\ &\quad (1, 0, 1), (1, 1, 2), (1, 2, 3), (1, 3, 4), \dots \\ &\quad (2, 0, 2), (2, 1, 3), (2, 2, 4), (2, 3, 5), \dots \\ &\quad \vdots \quad \quad \quad \}. \end{aligned}$$

In fact, we see only two ways out of this situation: First, we appeal to the common platonism in mathematics and stipulate that $+$ exists in the structure \mathfrak{N} in the way we expect it and with the properties we want it to have. Or, secondly, we treat $+$ as a proper set-theoretic object and construct it within our favorite axiomatic set theory, let's say ZFC.

While the second one is a solution concerning the level of Arithmetic, it simply shifts the problem to set theory. Within ZFC, the defined $+$ would be a red object (relative to ZFC), and we would have to ask now, how the models of ZFC look like—which is, not surprising, a much harder question than to look for structures for Arithmetic, not only in a technical sense, but also in a philosophical one; in particular, it leads us simply back to the situation we are discussing for $+$ just for other, now set-theoretical, objects.

For the first option, the sheer existence of $+$ presupposes a certain form of platonism. This platonism allows to dispose of any further technical questions, but one enters a dangerous philosophical terrain.

We may leave aside here the preferred way out (this is a philosophical question—but one which is naturally asked in this context and can nicely be brought out and talked about with the help of our colours); what we consider as an important lesson for the teaching of logic is, that any discussion of the syntax/semantics distinction in the context of Arithmetic—and, *a fortiori*, in relation to Gödel's Incompleteness Theorems—has to presuppose some “blue objects”. And this presupposition is far from being trivial. As a minimal conclusion, we like to state that an unreflected “identification” of, for instance, the “syntactical addition” (symbol) $+$ with the “semantical addition” (function) $+$ prevents a student from any understanding of the problem which is at issue here.

4 Gödel's Incompleteness Theorems

Gödel's Incompleteness Theorems are, without doubt, the most important results in mathematical logic. They are, however, clouded by continuing philosophical misunderstandings,⁷ and a pure formal presentation of the results can easily support such prejudices.⁸

We like to address only one problem coming from the fact that many presentations make use of the *Chinese remainder theorem*. Apparently, here proper mathematics is used to prove something about formal systems for mathematics and one may wonder whether there is some kind of vicious circle in the back.⁹ This is, of course, not the case; in fact, the use of this theorem is only relevant for a certain technical step, in particular in the case of Peano

⁷ For a detailed discussion, in particular in view of Gentzen's later results, we refer to [10].

⁸ An extreme case was reported from a philosophical seminar: the participants, after studying carefully—and probably painfully—a formal presentation of Gödel's proof, came to the conclusion that, yes, after all the results seem to be correct *if one has a language with 7 symbols; but, of course, one doesn't know whether it still holds when one would have 8 symbols*.

⁹ The first author remembers an elaboration of a Math student of a course presentation of Gödel's theorems which focussed nearly entirely on the use of the Chinese remainder theorem; apparently it was the only part of the proof the student was properly understanding based on his Math classes.

Arithmetic, and it is irrelevant for other axiomatization or codings (see [16, §3.2.6]). Hao Wang reported this from interviews with Gödel [17, p. 653]: “He enjoyed much the lectures by Furtw[ä]ngler on number theory and developed an interest in this subject which was, for example, relevant to his application of the Chinese remainder theorem in expressing primitive recursive functions in terms of addition and multiplication.” Thus, what is at issue here is the representation of primitive recursive functions in the formal arithmetical theory. Albeit, it is, of course, possible to develop primitive recursion *in* such theories, a proper educational approach *can* “outsource” primitive recursion.

We did that by setting up a new realm for primitive recursion (formally a functional algebra, consisting of function symbols and equalities)—and using a new colour for it, as, for the moment, it can be located outside of the red and blue world. After providing a sufficient stock of primitive recursive functions, one simply proves the representation theorem showing that every such function can be represented in the formal theory under consideration (as Peano Arithmetic, for instance). In the particular case of Peano Arithmetic, this proof may use the Chinese remainder theorem—but it is obvious that it is not relevant if one would provide another proof, in particular for other theories. In addition, it should become clear that Gödel’s theorems can be carried out in essentially the same way for all theories allowing for the appropriate representation theorem.

With respect to the Chinese Remainder Theorem and the like the diagnosis is as following: The Chinese Remainder Theorem is—as far as Gödel’s first incompleteness theorem is concerned—at the level of meta theory, but not necessarily in the blue world. It deals with the recursion-theoretic realm, concerning codings, and can be outsourced and imported if needed.

The situation changes however, if one wants to proceed to the second Incompleteness Theorem: There one repeats the proof of the first completely inside the red theory. For that case it is of vital importance that one uses codings and theorems that can be reproduced in the object theory itself—this is the sense in which it is “occasionally not just important [in logic] what you prove, but how you prove it” [12, p. 190].

5 “Semantics comes first”?

The dictum “Semantics comes first” is attributed to Tarski.¹⁰ And it has some rationale: in first-order logic as well as in Arithmetic, we presuppose the *meanings* of the logical and arithmetical expressions before we start to manipulate them. It is, in general, seen as a task of the axiomatization to provide calculi which capture such meaning—and the completeness theorems should show us that the axiomatization was successful.

As much as completeness theorems are concerned, they give us—*a posteriori*—the possibility to identify the red with the blue objects. But, to make sense out of a completeness theorem—and to prove it—the difference of the syntactical and semantical expression has to be seen clearly. In this perspective, we could even propose that one could forget of the red and blue colour for, let say, equality, *after* the proof of the completeness theorem for first-order logic.

But there is still an issues to discuss which might call Tarski’s dictum in question.

First, the dictum makes sense only, if the semantics is *understood before* the syntax. It is doubtful that semantics could serve its purpose if it is, by itself, not fully understood. For

¹⁰ The well-known computer software package *Tarski’s World* [2] illustrates this slogan quite appropriately when the student can literally manipulate the semantical objects.

first-order logic and Arithmetic it should be the case that their semantics is clear (although we mentioned the problem to precisely articulate it in the case of $+$, for instance). And this holds probably also for most of the mathematical theories we are familiar with.

The situation changed drastically, when Computer Science entered the stage. Semantics of programming languages is a rather challenging topic in CS. And here, the syntax is essentially always prior to the semantics. Instead of finding an axiomatization for a given semantics, now one looks for a semantics for a given programming language. But it is not only that such a semantics is no longer “first”, it might be the case that proposed semantics are, in a technical sense, more complicated than the programming language in itself (as example, we may refer to [1]). In some cases, one might wonder in which sense such semantics provide any *meaning*.¹¹

From the educational perspective, this only means that logic cannot any longer be taught with Tarski’s dictum as a rock solid starting point. The differentiation of syntax and semantics becomes even more significant. What it means to put “syntax first” we will discuss in the next section.

6 Intuitionism

When Brouwer conceived his mathematical philosophy of *intuitionism* he had definitely nothing formal in mind. However, based on the “successful” axiomatization of intuitionistic logic by Heyting, today, intuitionism is often presented in an axiomatic context; for propositional logic, for instance, just like a calculus for classical logic, but without *tertium-non-datur*. In this way, it can be recast totally in our red world. An “additional” semantics, may it be informal or set-theoretically, would be alien to intuitionism.¹² One could go a step further and say—instead of that there is no blue world—that the red world should be its own blue world. This is often paraphrased by saying that, from an intuitionistic perspective, the meaning of a formula is given by its proof (better: the set of all its proofs). Today, this idea is reflected in *proof-theoretic semantics*, an approach which “attempts to locate the meaning of propositions and logical connectives not in terms of interpretations . . . but in the role that the proposition or logical connective plays within the system of inference.”¹³

In view of our discussion above, the formal framework of intuitionism undermines substantially Tarski’s dictum. In particular, when we go back to Definition 1 of a L-structure, one can easily observe that clause (iii) for negation *builds in* classical logic into any structure.

It is worth to reflect a little bit more on this clause; it looks as innocent as any other clause, that for conjunction and disjunction, for instance, “just making sure that each of our symbols means what we want it to mean”, to repeat the citation from Barwise [3, p. 21]. But it is not only the case, that Brouwer would not agree that this is what we want negation to mean; it is the problem that the right-hand side—not $\mathfrak{M} \models \varphi[s]$ —can, in general, not practically be verified. It is the full purpose of an axiomatization to give a *positive* approach to negation, i.e., the symbol for negation \neg has to stay always on the right-hand side of the

¹¹ A colleague expressed this in a sarcastic statement: “They put these brackets $[\cdot]$ around something and believe they have achieved something.”

¹² The first author remembers a remark of an author to the effect that intuitionism was not succeeding due to the lack of a proper semantics, until this deficit was cured by the introduction of Kripke semantics. In our view, this sounds like claiming that a protestant church is not a church as long as it doesn’t have a pope.

¹³ http://en.wikipedia.org/wiki/Proof-theoretic_semantics, accessed Febr. 20, 2015. For more on proof-theoretic semantics, see [11, 13].

derivation sign \vdash , and we must not make use of a “meta-negation” \neg . That this is possible is far from being trivial—the complexity is visible in the proof of Gödel’s completeness theorem—but it makes clear the fundamental difference between syntax (in form of an axiomatization, providing an—executable, albeit not decidable—derivation relation) and semantics, which is, in general, inherently non-constructive.

As much as it comes to Brouwer’s criticism of the usual semantic understanding of negation, it is worth to cite Bernays [4, p. 4 (our translation)], who replied to it as follows:

As one knows, the use of the “tertium-non-datur” in relation to infinite sets, in particular in Arithmetic, was disputed by L. E. J. BROUWER, namely in the form of an opposition to the traditional logical principle of the excluded middle. Against this opposition it is to say that it is just based on a reinterpretation of the negation. BROUWER avoids the usual negation non-A, and takes instead “A is absurd”. It is then obvious that the general alternative “Every sentence A is true or absurd” is not justified.

Following Bernays, the controversy about classical and intuitionistic negation could even be boiled down to a question of words. What is of importance for us here, is that the semantic determination of (classical) negation closes the road to understand Brouwer’s (intuitionistic) negation. To say it differently: the “standard approach” of teaching logic based on Tarski’s dictum (used for Tarskian semantics as given in Def. 1) blocks a proper transition to intuitionistic logic, at least, if the students are “semantically biased”. As a matter of fact, intuitionistic logic can—and maybe: has to—be developed purely in the red world.¹⁴ It is not our aim to motivate an approach that exclusively focusses on intuitionism or the like.¹⁵ But it is important that an introduction to classical logic should not come with a semantical bias which blocks the road to non-classical logic from the start.¹⁶ Here the colours may help.

7 Conclusion

It is our experience from courses on mathematical logic—regular university classes, block courses, and summer school courses—that the use of colours gives the students, indeed, a “tool” at hand to see better the fundamental distinction of syntax and semantics.¹⁷ Although the use of colours requires some additional effort in the preparation of the student material, it is a rather inexpensive investment to obtain significant educational added value. At a certain stage—for instance, as mentioned, after the proof of the completeness theorem—, when the student has already internalized the difference of syntax and semantics one can even easily go back to the common identification of the syntactical and semantical expression using them just in black; it provides even a good test template, as one can ask the students in all kind of instances whether a certain expression would have to be coloured red or blue.

¹⁴ As said at the beginning, this concerns intuitionistic logic based on Heyting’s axiomatization; Brouwer’s original intuitionism, in particular if considered in the mathematical area of Analysis, should not come in a formal livery at all.

¹⁵ It is worth mentioning that *dialogical logic* has a rather interesting status concerning the syntax and semantics distinction, which doesn’t seem to be properly explored yet, see Kahle [8, 9].

¹⁶ For instance, substructural logics are best approached via proof theory, because—quite like intuitionism—there lies their motivation; on the other hand their semantics usually turn out to be rather complicated.

¹⁷ We have, of course, not developed an empirical study to “prove” that the use of colours improves the teaching; it is rather an informed impression backed by feedback from our students. And although we do not have statistically valid data comparing the effects of this method with other approaches, the test results of the courses were very positive (but we had few, but rather good and understanding students).

In addition, while the colouring can, of course, not solve any philosophical problem concerning syntax and semantics, it helps to sharpen the sensibility for the underlying problems. In any case, the success of a logic course will not only depend on the use of tools, but on the interaction between teacher, tutor(s) and students. The use of a new dimension—like colour—underlines the importance of the distinction between syntax and semantics, it brings various (philosophical) topics into focus and gives a convenient way of talking about these issues—a way that often is readily accepted by the students.

The message of this paper is not: By all means use colours to distinguish syntax from semantics! But rather: Distinguish syntax and semantics clearly and discuss some of the issues related to that distinction—the use of colour will help you doing so!

References

- 1 Th. Altenkirch, P. Morris, F. Nordvall Forsberg, and A. Setzer. A categorical semantics for inductive-inductive definitions. In A. Corradini and B. Klin, editors, *Algebra and Coalgebra in Computer Science*, volume 6859 of *Lecture Notes in Computer Science*, pages 70–84. Springer, 2011.
- 2 D. Barker-Plummer, J. Barwise, and J. Etchemendy. *Tarski's World*, volume 169 of *CSLI Lecture Notes*. CSLI Publications, 2008.
- 3 J. Barwise. An introduction to first-order logic. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 5–46. North-Holland, 1977.
- 4 P. Bernays. Bemerkungen zu LORENZEN's Stellungnahme in der Philosophie der Mathematik. In K. Lorenz, editor, *Konstruktionen versus Positionen*, volume 1, pages 3–16. Berlin, 1979.
- 5 R. Cori and D. Lascar. *Mathematical Logic, Part I*. Oxford University Press, 2000.
- 6 B. Buldt et al., editor. *Kurt Gödel. Wahrheit und Beweisbarkeit*, volume 2: Kompendium zum Werk. öbv & hpt, 2002.
- 7 S. Feferman. Kurt Gödel: Conviction and Caution. *Philosophia Naturalis*, 21:546–562, 1984.
- 8 R. Kahle. Konstruktivismus und Semantik. In J. Mittelstraß, editor, *Der Konstruktivismus im Ausgang der Philosophie von Wilhelm Kamlah und Paul Lorenzen*, pages 197–212. Mentis, 2008.
- 9 R. Kahle. Dialoge als Semantik. In J. Mittelstraß and Chr. von Bülow, editors, *Dialogische Logik*, pages 43–54. Mentis, 2015.
- 10 R. Kahle. Gentzen's consistency proof in context. In R. Kahle and M. Rathjen, editors, *Gentzen's Centenary*. Springer, 2015, to appear.
- 11 R. Kahle and P. Schroeder-Heister, editors. *Proof-theoretic semantics*, volume 148. 2006.
- 12 W. Rautenberg. *A Concise Introduction to Mathematical Logic*. Springer, 2006.
- 13 P. Schroeder-Heister. Proof-theoretic semantics. In E.Ñ. Zalta, editor, *The Stanford Encyclopedia of Philosophy*. Summer 2014 edition, 2014.
- 14 A. Sernadas and C. Sernadas. *Foundations of Logic and Theory of Computation*, volume 10 of *Texts in Computer Science*. College Publication, 2008.
- 15 J. R. Shoenfield. *Mathematical Logic*. Addison-Wesley, Reading, MA, 1967. Reprinted: Association for Symbolic Logic and AK Peters, 2001.
- 16 C. Smoryński. The incompleteness theorems. In J. Barwise, editor, *Handbook of Mathematical Logic*, pages 821–865. North-Holland, 1977.
- 17 H. Wang. Some facts about Kurt Gödel. *The Journal of Symbolic Logic*, 46:653–659, 1981.

